MATH2050C Selected Solution to Assignment 12

Section 5.4 no. 3, 4, 6-12.

(3) (a) $f(x) = x^2$ is not uniformly continuous on $[0, \infty)$. Pick $a_n = n$ and $b_n = n + 1/n$. Then $|a_n - b_n| = 1/n \to 0$ but $|f(a_n) - f(b_n)| = 2 + 1/n^2 > 2$.

Note. In general, any polynomial of degree ≥ 2 is not uniformly on any unbounded interval. (Of course, it is uc on every bounded interval.)

(b) $g(x) = \sin 1/x$ on $(0, \infty)$. Pick $a_n = 1/(2n\pi)$ and $b_n = 1/(2n + 1/2)\pi$. Then $|a_n - b_n| \to 0$ but $|\sin 1/a_n - \sin 1/b_n| = |0 - 1| = 1$ for all n.

(4) Let us prove a more general result. Let f be a continuous function on $[0, \infty)$ which satisfies $\lim_{x\to\infty} f(x) = 0$. Then f is uniformly continuous on $[0, \infty)$. For, given $\varepsilon > 0$, there is some $K > 1$ such that $|f(x)| < \varepsilon/2$ for all $x \in [K, \infty)$. On the other hand, as f is continuous on $[0, K + 2]$, it is uniformly continuous there. We can find some $\delta < 1$ depending only on ε such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$, $x, y \in [0, K + 1]$. Now, if $x_0 \in [0, K]$, $|x - x_0| < \delta$ implies $x \in [0, K + 1]$, so $|f(x) - f(x_0)| < \varepsilon$. If $x_0 \in [K + 1, \infty)$, for x satisfying $|x - x_0| < \delta < 1$, $x \in [K, \infty)$, hence $|f(x) - f(x_0)| \leq |f(x)| + |f(x_0)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, done.

(6) Let f be bounded by M and g by K. Use

$$|f(x)g(x) - f(y)g(y)| = |(f(x) - f(y))g(x) + f(y)(g(x) - g(y))| \leq K|f(x) - f(y)| + M|g(x) - g(y)| .$$

(7) The functions x and $\sin x$ are uniformly continuous on $(-\infty, \infty)$, but its product $h(x) = x \sin x$ is not. Let $a_n = 2n\pi$ and $b_n = (2n + 1/n)\pi$ so $|a_n - b_n| \to 0$. On the other hand,

$$\frac{\sin\left(2n\pi + \dfrac{1}{n}\pi\right)}{\pi/n} = \frac{\sin\dfrac{\pi}{n}}{\pi/n} \to 1 , \quad \text{as } n \to \infty .$$

Thus,

$$|b_n \sin b_n - a_n \sin a_n| = |b_n \sin b_n| \to 2\pi^2 , \qquad \text{as } n \to \infty.$$

(8) Same as the proof of the composite of two continuous functions is continuous, just noting that δ depends on ε only.

(10) If not, there is a sequence $\{x_n\}$ in A such that $|f(x_n)| \geq n$. As A is bounded, by Bolzano-Weierstrass, by passing to a subsequence if nec, we may assume $x_n \to x^*$ for some x^* (not nec in A). Then $\{x_n\}$ is a Cauchy sequence. Now, by assumption f is uniformly continuous, for $\varepsilon = 1$, there is some δ such that $|f(x) - f(y)| < 1$ whenever $|x - y| < \delta$. As $\{x_n\}$ is a Cauchy sequence, $|x_n - x_m| < \delta$ for all $n, m \geq n_0$. But then

$$n \leq |f(x_n)| \leq |f(x_n) - f(x_{n_0})| + |f(x_{n_0})| \leq 1 + |f(x_{n_0})| ,$$

which is impossible for large n. Hence, f must be bounded.

(15) (c) An example is the linear function $f(x) = x$. Clearly it is Lipschitz continuous, but x^2 is not.

Supplementary Exercise

1. Let f be continuous on (a,b), $-\infty \le a < b \le \infty$. Show that it is uniformly continuous on (a,b) if it is uniformly continuous on $(a,c]$ and $[c,b)$ for some $c \in (a,b)$.

   Solution. For $\varepsilon > 0$, we fix some δ such that $|f(x)-f(y)| < \varepsilon/2$ for $x,y \in (a,c], |x-y| < \delta,$. Also, we fix δ' that $|f(x)-f(y)| < \varepsilon/2$ for $x,y \in [c,b), |x-y| < \delta'$. We let $\delta_1 = \min\{\delta, \delta'\}$. Let $x \in (a,c]$ and consider $y \in (x-\delta_1, x+\delta_1)$. If y also belongs to $(a,c]$, using $\delta_1 \le \delta$, we have $|f(x)-f(y)| < \varepsilon/2 < \varepsilon$. If $y \in (c,b)$, observe that $|x-c|, |c-y| < \delta_1$ and so $|f(x)-f(y)| \le |f(x)-f(c)| + |f(c)-f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Similarly, we handle $x \in [c,b)$.

2. Consider $h(x) = 1/x$. Show that it is continuous on $(0,1]$ by determining the best δ as a function of ε and x_0. And then using it to show h is not uniformly continuous on $(0,1]$ but uniformly continuous on $[a,1]$ for any fixed $a \in (0,1)$. (This was done in class.)

3. Optional. Consider $g(x) = x^{-2}$. Show that it is continuous on $(0,\infty)$ by determining the best δ as a function of ε and x_0. And then using it to show g is not uniformly continuous on $(0,\infty)$ but uniformly continuous on $[a,\infty)$ for any fixed $a > 0$.

   Solution. Let $x_0 \in (0,\infty)$. We determine $x_1 < x_0 < x_2$ so that $[x_1,x_2]$ is mapped to $[g(x_0)-\varepsilon, g(x_0)+\varepsilon]$. Since g is strictly decreasing, we know that $g(x_0)-\varepsilon = g(x_2)$ and $g(x_0)+\varepsilon = g(x_1)$. By solving the equations we get

   $$x_1 = \frac{x_0}{\sqrt{1+\varepsilon x_0^2}}\ , \quad x_2 = \frac{x_0}{\sqrt{1-\varepsilon x_0^2}}\ .$$

   From $x_2 - x_0 < x_0 - x_1$ we find that the best δ is given by $x_0 - x_1$:

   $$\delta(\varepsilon, x_0) = x_0 - x_1 = \frac{\varepsilon x_0^3}{\sqrt{1+\varepsilon x_0^2}\ (1+\sqrt{1+\varepsilon x_0^2})}\ .$$

   As $x_0 \to 0$, $\delta(x_0,\varepsilon) \to 0$. Therefore, g is not uniformly continuous on $(0,\infty)$.

   Next, we are going to show that g is uniformly continuous on $[a,1]$ and $[1,\infty)$. By the previous problem, it is uniformly continuous on $[a,\infty)$. For $x_0 \in [a,1]$, we have

   $$\delta(x_0,\varepsilon) \ge \frac{\varepsilon a^3}{\sqrt{1+\varepsilon}(1+\sqrt{1+\varepsilon})} \equiv \delta_1(\varepsilon)\ .$$

   It follows that $|g(x)-g(x_0)| < \varepsilon$ on $[a,1]$ whenever $|x-x_0| < \delta_1$. Next, for $x_0 \in [1,\infty)$,

   $$\begin{aligned} \delta(x_0,\varepsilon) &= \frac{\varepsilon x_0}{\sqrt{x_0^{-2}+\varepsilon}\left(x_0^{-1}+\sqrt{x_0^{-2}+\varepsilon}\right)} \\ &\ge \frac{\varepsilon}{\sqrt{1+\varepsilon^{1/2}}(1+\sqrt{1+\varepsilon})} \\ &\equiv \delta_2(\varepsilon)\ , \end{aligned}$$

   Note. Not insisting on using the ε-δ thing, it suffices to consider $a_n = 1/n, b_n = 2/n \to 0$ but $g(1/n) - g(2/n) = 3n^2/4 \to \infty$ as $n \to \infty$. It shows that g is not uniformly continuous in any interval of the form $(0,a), a > 0$.

4. Optional. Let E be a non-empty set in $\mathbb{R}$. Define the distance function $\rho(x) = \inf\{|z-x| : z \in E\}$. Show that

$$|\rho(x) - \rho(y)| \leq |x - y|.$$

Solution. For all $z \in E$, $\rho(x) \leq |z - x| \leq |z - y| + |y - x|$. Taking infimum over $z \in E$ on the right hand side, we get

$$\rho(x) \leq \rho(y) + |y - x| \ ,$$

and the result holds in view of the symmetry between x and y.

Note. When E is a point set $\{x_0\}$, the distance function $\rho(x) = |x - x_0|$. It is differentiable except at $\{x_0\}$. In general, the distance function can be defined for any subset in $\mathbb{R}^n$ in a similar way. It turns out the same proof establishes its Lipschitz continuity. It shows Lipschitz continuity is a fundamental concept in analysis. A deep theorem of Radamarcher asserts that every distance function is differentiable at most points.